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A Study of Hierarchical Risk Parity in Portfolio Construction

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ABSTRACT

The construction and optimization of a portfolio is a complex process that has been a historically active research area in finance. For portfolios with highly correlated assets, the performance of traditional risk-based asset allocations methods such as the mean-variance (MV) model is limited when numerous assets are correlated. A novel clustering-based asset allocation method, called Hierarchical Risk Parity (HRP), provides an opportunity to mitigate these limitations in portfolio construction. HRP utilizes the hierarchical relationships between the covariance of assets in a portfolio to determine weight distributions, eliminating the need for the inversion of the covariance matrix that is required by most traditional risk-based asset allocation methods. This research examines the viability of Hierarchical Risk Parity (HRP) method in portfolio construction of a US equity portfolio and compares the performances of HRP to traditional asset allocation methods exemplified by the mean-variance (MV) method. The results of this research show that the performance of the HRP method is comparable to the performance of the MV method. Given these findings, HRP provides an advantageous approach for portfolio construction in practical scenarios where correlated assets are present in the portfolios.

KEYWORDS

Hierarchical clustering; hierarchical risk parity; mean-variance method; portfolio construction; machine learning in finance

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1. Introduction

Portfolio optimization is both a historic and current area of importance in quantitative finance. A portfolio is a collection of financial assets such as bonds, stocks, cash or other financial assets such as exchange-traded funds (ETFs), mutual funds etc. The core of the portfolio is composed of fixed-income such as bonds along with equities or stocks, and cash or cash equivalents. In asset allocation, investments are divided among the assets present in the portfolio with the aim of balancing the risk and rewards of the portfolio. In portfolio optimization investments are allocated to assets such that the return on investment is maximized while the risk is minimized.

Markowitz in 1952 had developed the portfolio optimization technique using the mean-variance optimization framework (Markowitz, 1952). Mean-variance model is the base of the Modern Portfolio Theory (Hult et al., 2012). According to the mean—variance (MV) theory, the investment weights among the assets are distributed depending upon an optimal balance between mean and variance of the portfolio. Mean of the portfolio is a measure of its profit and variance of the portfolio is a measure of its risk. Thus, mean-variance analysis deals with weighing risk against return of the portfolio. Markowitz's mean-variance model though optimal in theory faces two primary challenges—(1) the model requires a forecasting of returns which is prone to errors, (2) the model also requires an inversion of the covariance matrix which can make the portfolio unstable if the inversion of the covariance matrix is ill-conditioned which occurs when correlated assets are present in the portfolio.

In an attempt to overcome the first problem of the modern portfolio theory, researchers examined how to formulate the risk-based optimization strategies that will not require the estimation of returns and subsequently developed strategies such as the Equal Weight method (Maillard et al., 2010; Bruder and Roncalli, 2012). The risk-based asset allocation methods though do not require an estimation of the expected returns, inversion of the covariance matrix is still a requirement for the risk-based asset allocation methods which results in underperformance and instability of the portfolios with correlated assets (Lopez de Prado, 2018).

Hierarchical Risk Parity (HRP) approach is an optimization technique that was suggested by Lopez de Prado (2018). The HRP algorithm is a hierarchical-clustering based method that builds a portfolio using the covariances and correlations of the assets in the portfolio. HRP algorithm does not require an inversion of the covariance matrix. HRP is a robust portfolio allocation method that does not require forecasting of returns and applies machine learning techniques and graph theory (Lopez de Prado, 2018).

In this study, two portfolios with US equities have been constructed using two different methods-Hierarchical Risk Parity (HRP) method, which is a hierarchical clustering-based asset allocation method and mean-variance (MV) method, which is a traditional risk-based asset allocation method. The performances of the HRP method and mean-variance method have been compared. In practical portfolio construction it is customary to have correlated assets. It has been demonstrated in this study that a hierarchical clustering-based method such as the HRP can handle correlated assets in the portfolios without being challenged by risk concentration even when the condition number of the covariance matrix is high. This research asserts that HRP is a viable method for constructing portfolios with US equities. It has also been presented that portfolios constructed using HRP do not have risk concentration. In this research, it has been shown that the performance of the HRP method is comparable to the performance of the mean-variance method with oftentimes the HRP method outperforming the mean-variance method.

2. Literature Review

One of the most important advantages of Markowitz's theory is its intuitive nature and customizability (Roncalli, 2013). The mean-variance optimization follows the pareto-optimal in-sample allocation property (Kolm et al., 2014). The mean–variance model, however, has a few drawbacks that would restrict the application of the model to practical portfolio construction. Firstly, the model expects returns based on the mathematical calculations of the past data as an input which is prone to significant errors; and secondly, the method requires inversion of the covariance matrix to calculate the distribution of weights (Lopez de Prado, 2018; Steinbach, 2001; Rubinstein, 2002). The mean-variance model is an estimation error optimizer and it tends to underperform on the out-of-sample data because the model allots greater weight on the assets which had greater returns (Braga, 2016). The model also generates concentrated weight distribution and is prone to estimation errors (Demey et al., 2010; Green and Hollifield, 1992).

One of the commonly used risk-based asset allocation techniques derived from the mean-variance approach is the minimum-variance method (Markowitz, 1952). In the minimum-variance method the risk of the portfolio is minimized (Hult et al., 2012). A minimum-variance portfolio lies on the efficient frontier at a lower variance and is not constrained by any expected returns (Munk, 2018). The errors in the estimation of return are considerably higher than the errors associated in the estimation of the covariance matrix (Merton, 1980; Jagannathan and Ma, 2003). This has led to an increased interest in the risk-based asset allocation methods where investors do not need to estimate the returns (Maillard, 2010). In the minimum-variance method weight is allocated only on the assets with low volatility which might lead to the concentration of risk in the portfolio. The minimum-variance method is fairly sensitive to the estimation errors of the covariance matrix (Ardia et al., 2017).

The equal-weighted method is a simple asset allocation strategy in which all the portfolio assets receive the same amount of investment weight. There has been substantial empirical research work to suggest that in spite of the simplicity of the model, in many instances it has outperformed many sophisticated allocation strategies (DeMiguel et al., 2007). In practice, the equal-weighted portfolio is used widely (Windcliff and Boyle, 2004). The equal-weighted portfolio provides a balanced allocation of weights; however, if there is a significant difference in the risks of the assets in the portfolio, equal-weighted allocation might have a concentrated risk allocation which could be viewed as a disadvantage of the approach (Benartzi and Thaler, 2001).

Risk Parity method, also known as the equal risk contribution (ERC) method, is based on allocation of volatility among the assets in the portfolio. In the ERC method, each asset in the portfolio contributes equally to the overall risk of the portfolio. The ERC approach has been derived from a simple technique called risk budgeting (Scherer, 2007). This approach was first introduced by Bridgewater Associates in 1996 and the term was created by Qian (2005). Qian (2005) suggested that investment portfolios should have risk-allocation versus capital-allocation. Risk contribution from equities in the portfolio is more significant than the risk contribution from fixed income assets in the portfolio; and hence, during a financial downturn more loss would likely come from equities than from fixed income assets of the portfolio (Qian, 2005). Empirical data for this theoretical assumption showed that for a portfolio with 60% equity and 40% fixed-income, 90% of the risk contribution was from the equities (Illmanen and Villalon, 2012).

Hierarchical Risk Parity (HRP) is a hierarchical clustering-based asset allocation method. In unsupervised machine learning, clustering is the most common technique that involves grouping of the unlabeled data points based on their characteristics (Maimom and Rokach, 2005). Objects sharing similar features are clustered together and objects having dissimilar features are present in different clusters (Murtagh and Contreras, 2017). In hierarchical clustering the number of clusters does not need to be specified and the algorithm is deterministic in most cases. The output of hierarchical clustering is a structure that has hierarchy. At the end of the clustering method, the model provides a cluster ID for each cluster that gets created by the algorithm (Manning et al., 2019).

3. Methodology

3.1. Calculation of Portfolio Return

3.1.1. Calculation for the individual assets

Return (r_i) of an asset (i) in a portfolio (p) is calculated by subtracting the previous-day closing price (P_0) of the asset from the present-day closing price (P_1) and by dividing the result by the previous-day closing price (P_0) as shown below:

$$r_i = \frac{P_1 - P_0}{P_0} = \frac{P_1}{P_0} - 1 \tag{1}$$

In the variance-covariance matrix of the portfolio, the variance (σ^2) is present along the diagonal position and the covariance of the assets is present along the non-diagonal position. The variance-covariance matrix has been calculated on a one-year look back period assuming that one-year has 250 trading days. For N number of observations, variance of an asset can be expressed as below:

$$\sigma^2 = \frac{\sum_{i=1}^{N}(r_i - \bar{r})^2}{N} \tag{2}$$

where, i is the number of observations for an asset. For a one-year look back period, N = 250.

Covariance $(\sigma_{(X,Y)})$ indicates the direction of movement of the assets (X, Y) in the portfolio. In positive covariance both the assets move in the same direction at the same time i.e., both the assets are high or low at the same time. In negative covariance when one asset is high the other asset tends to be low i.e., the assets move in the opposite direction.

$$\sigma_{(X,Y)} = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{N} \tag{3}$$

Correlation of the assets $(\rho_{(X,Y)})$ indicates the strength of relationship between the assets in a portfolio. It shows the degree to which the two assets move in coordination to one another. Correlation coefficient ranges between -1.0 to +1.0.

$$\rho_{(X,Y)} = \frac{\sigma_{(X,Y)}}{\sigma_X * \sigma_Y} \tag{4}$$

3.1.2. Calculation for the portfolio

The constraint in weight distribution for a portfolio with N number of assets is presented below.

$$\mathrm{W} = \sum_{i=1}^{N} w_i = 1 \tag{5}$$

where, w_i represents weight on the asset i. Thus, the summation of the distributed weight on all assets is the total weight of the portfolio (W), which is equal to 1.

Return of the Portfolio (r_p) with N number of assets can be calculated by the summation of the product of the return of asset and weight on the asset; and can be expressed as,

$$r_p = \sum_{i=1}^{N} w_i \, r_i \tag{6}$$

where, r_i represents *returns* on each asset.

Variance of the portfolio (σ_P^2) indicates the risk of the portfolio and can be expressed as the variance of return of the portfolio (r_p) as shown below. A portfolio with a higher variance is a portfolio with greater risk.

$$\sigma_P^2 = Var\,(r_p) \tag{7}$$

The variance of the portfolio (σ_P^2) can also be expressed in terms of covariance of the assets as shown below.

$$\sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \, w_j \, \sigma_{i,j} \tag{8}$$

where, $i \; and \; j$ are the assets in the portfolio, and $\sigma_{i,j}$ is the covariance between assets $i \; and \; j$.

3.2. Optimal Risk Portfolio

The process of efficient allocation of capital among a group of assets in the portfolio is known as optimal portfolio construction (Maillard et al., 2010). Optimal or efficient portfolio yields high returns with a certain amount of risk which an investor agrees to take. Quantitative modeling is one of the effective ways in creating an efficient portfolio.

Markowitz's (1952) Mean-variance (MV) method is a risk based quantitative model which is the foundation of the Modern Portfolio Theory. Mean-variance method suggests that a trade-off relationship exists between return and risk of a portfolio (Markowitz, 1952). According to the MV method investors maximize the return of a portfolio for a given level of risk or volatility. Optimal portfolios lie on the Markowitz Efficient Frontier curve. The X-axis of the Efficient Frontier plot represents the standard deviations of the portfolios and the Y-axis represents the expected returns. The portfolios that lie below the efficient frontier curve are sub-optimal and the portfolios that lie on the efficient frontier curve are the optimal portfolios. The portfolios that are present on the leftmost point of the curve exhibit the lowest variance. Optimal portfolios have maximum Sharpe Ratio. Sharpe ratio is the ratio between return and risk of the portfolio (Sharpe, 1964).

The distribution of weight among the assets in the portfolio is calculated from the quadratic equation (8). To calculate the distribution of weight from the quadratic equation (8), an inversion of the covariance matrix takes place.

3.3. Designing the portfolio for Hierarchical Risk Parity

Hierarchical Risk Parity (HRP) uses the information present in the covariance matrix to calculate the distribution of weights among the assets in the portfolio. HRP works on hierarchical structures which impart stability and are intuitive as well.

The Hierarchical Risk Parity algorithm operates in three stages: Tree Clustering, Quasi-diagonalization, and Recursive bisection.

Tree Clustering: Tree Clustering is the first stage of the algorithm. In this stage assets are combined using agglomerative clustering. A cluster can have a single asset or multiple assets. When there are two constituents in a cluster, one constituent is referred to as a left child and the other as a right child. With N number of assets, N-1 number of clusters can be formed.

Assets (i and j) with similar correlations are considered close to one another and are clustered together. The distance ($d_{i,j}$) between the assets, i and j, can be defined in term of their correlation ($\rho_{i,j}$) as,

$$d_{i,j} = \sqrt{0.5\,(1 - \rho_{i,j})} \tag{9}$$

Thus, a distance matrix could thus be created from the correlation matrix.

The value of the correlation coefficient ranges between -1 to +1. A perfect positive correlation has a correlation coefficient value of +1. Perfectly correlated assets would thus have a distance of 0. Perfect positive correlation between the assets means that when one asset moves up the other asset also moves in the same direction.

A perfect negative correlation between the assets has a correlation coefficient value of -1. A perfect negative correlation between the assets implies that the assets move in the opposite directions i.e., when one asset moves up the other asset moves down. When the value of the correlation coefficient between the assets is 0, no linear relationship exists between the assets.

When there are two assets (X and Y) in the portfolio that are not correlated to each other but in turn both display a similar correlation to another asset (Z) in the portfolio, the two assets (X and Y) could still be clustered because they could be thought of as assets exhibiting similar trends (Lopez de Prado, 2018). In this scenario, the distance between such clusters could be calculated as,

$$\bar{d}_{(i,j)} = \sqrt{\sum_{n=1}^{N} (d_{(n,i)} - d_{(n,j)})^2} \tag{10}$$

Thus, the Euclidean distance between the assets (i and j) is the distance between the columns i and j in the distance matrix.

The next step in the tree clustering is to recursively form the clusters of the assets. This would create a tree-like structure where each cluster is made up of subclusters which were clustered because of their closeness/similarity. The entire clustering process is stored in a linkage matrix.

Suppose, there are N numbers of assets which could be represented as *0, 1, ..., (N-1).* The linkage matrix would have a size of *(N-1) x 4*. Each row in the linkage matrix (L) represents a cluster in the clustering process and it could be represented as,

$$L = (y_{m,1}, y_{m,2}, y_{m,3}, y_{m,4}) \;\; for\; m = 1, 2, 3, \dots, (N-1) \tag{11}$$

The above row in the linkage matrix shows that $y_{m,1}$ and $y_{m,2}$ are the two clusters that were clustered together and, $y_{m,3}$ represents the distance between the said clusters. $y_{m,4}$ represents the number of original clusters (assets) that the resultant cluster would have. The input of the tree clustering in the hierarchical clustering algorithm is the covariance matrix and the output of the algorithm is the Linkage matrix (L).

Quasi-Diagonalization: During the step of Quasi-Diagonalization, the assets in the correlation matrix are reorganized so that the largest correlations lie around the diagonal. During this step, the similar assets are placed closer to each other, and dissimilar assets are placed further apart (Lopez de Prado, 2018). The input of the Quasi-diagonalization algorithm is the Linkage matrix (L) and the output of the algorithm is a Sorted List of the assets in the portfolio.

Recursive Bisection: In recursive bisection, the decision regarding allocation of weights to the assets in the portfolio takes place using inverse allocation variance through a tree structure. Initially all assets in the portfolio are allocated unit weight. The clustering tree represented in the form of a dendrogram is explored from top to bottom. Whenever bisection into two clusters takes place, a competition for weight would happen between the two resultant clusters. During this process, the volatility of each of the resultant clusters is computed and the assigned weight to the cluster is inversely proportional to its volatility. The input of the algorithm is the covariance matrix and a sorted list of assets and the output of the algorithm is the portfolio weight distribution. The input and out of the three stages of the algorithm has been summarized in Table I.

Table 1. Summary of the input and output of the HRP algorithm.

	Name	Input	Output
Stage 1	Tree Clustering	Covariance Matrix	Linkage Matrix
Stage 2	Quasi-Diagonalization	Linkage Matrix	Sorted List of Assets in the Portfolio
Stage 3	Recursive Bisection	Covariance matrix and sorted list of assets	Portfolio with distributed weights

4. Results

In this study, a clustering-based method has been applied in constructing a portfolio with US equities (S&P 500 sector ETFs) in order to demonstrate how hierarchical clustering rearranges covariance matrices for portfolio construction.

Presently, there are 11 sectors in S&P 500. In 2000, the number of sectors in S&P 500 was 9. Due to the availability and continuity of data, the portfolio has been constructed using the 9 sectors of S&P 500 that have been present since 2000.

Two portfolios with 9 sectors of ETFs (Exchange Traded Fund) price data of S&P 500 were built using MV model, which is a quadratic equation-based model and HRP model, which is a hierarchical clustering-based model. The performance of the two portfolios have been compared.

For this study, the closing price variable, 'Close', was selected for analysis. Daily closing price data for the 9 sector ETFs from January 31, 2000 to February 29, 2016 was procured from yahoo finance. The data from January 31, 2000 to February 29, 2016 was chosen in an attempt to keep a similar range of rising and falling markets. The conclusion and claim of this paper do not depend on the range of the market data selected.

Table 2 summarizes the sectors of S&P 500 whose ETFs were used to construct the portfolio for the research study. The price variation of the 9 sectors can be seen in figure 1.

Table 2. List of the Sectors of S&P 500 used for the analysis.

	ETF Ticker Symbols	Name of the Sectors
1	XLY	Consumer Discretionary
2	XLP	Consumer Staples
3	XLE	Energy
4	XLF	Financials
5	XLV	Healthcare
6	XLI	Industrials
7	XLB	Materials
8	XLK	Technology
9	XLU	Utilities

Figure 1 presents the historical closing prices of the 9 sectors of S&P 500. From the closing prices, daily returns of the sectors have been calculated using equation (1).

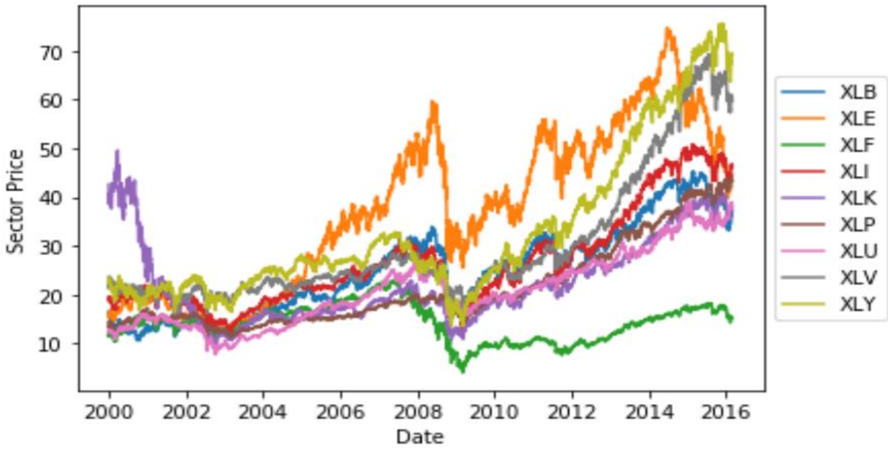

Figure 1. Historical sector ETF prices of S&P 500.

At the end of each month the covariance and correlation matrix has been calculated based on a one-year look back period on the closing price of the sector ETFs. The covariance and correlation matrix, generated at the month end, serves as an input in the MV algorithm and HRP algorithm.

As an example, in figure 2 the heatmap of the unsorted correlation matrix generated on the 29th of February 2000 has been presented that serves as an input in the HRP algorithm. Depending upon the correlation between the assets the HRP algorithm rearranges the correlation matrix to form clusters of the correlated assets as can be observed in the heatmap presented in figure 3, where the clusters of the correlated assets can be seen.

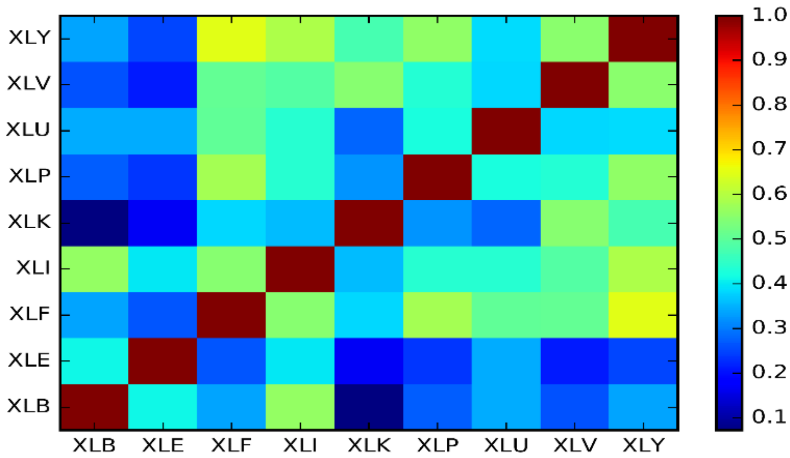

Figure 2. Heatmap of the unsorted correlation matrix on 2/29/2000.

The unsorted correlation matrix that was generated on the 31st of January 2012 has been presented in a heatmap in figure 4. The sorted correlation matrix by the HRP algorithm for the 31st January 2012 has been presented in a heatmap in figure 5.

The final output of the MV algorithm and HRP algorithm is the allocation of weight on the assets present in the portfolios. The weight allocation is calculated at the end of each month. The allocated weight on the portfolio assets at the end of the month is kept the same for the following month. From the portfolio weight distribution, daily return of the portfolio for the MV model and HRP model has been calculated using the equation (6).

Figure 6 and figure 7 shows the difference in weight allocation among the assets of the two portfolios that have been created using the MV model and HRP model. For example, in figure 6, it can be observed that on the 29th of

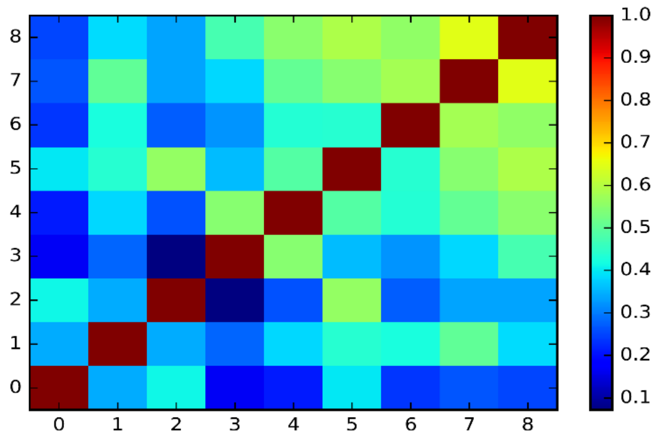

Figure 3. Heatmap of the sorted correlation matrix on 2/29/2000.

February, 2000 the MV model did not allocate any weight on XLF and XLY assets of the portfolio. Figure 7 shows that on the 31st of January, 2012 the MV model allocated weight only on XLP and XLU assets, which might lead to a risk concentration in the portfolio. On the other hand, it can be observed from figure 6 and figure 7 that in the portfolio created by the HRP algorithm weight was distributed on all the assets; thus, chances of risk concentration was much less as opposed to the MV model portfolio.

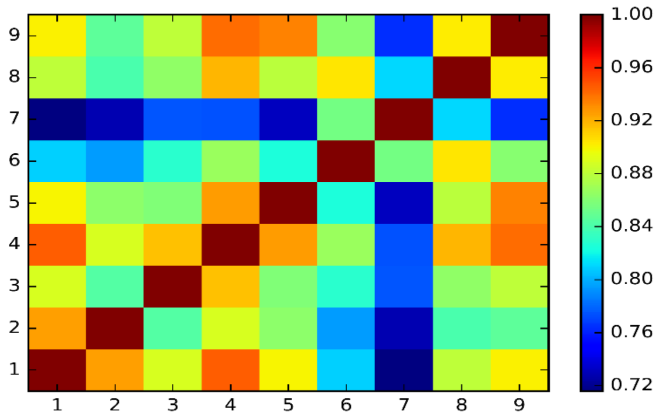

Figure 4. Heatmap of the unsorted/disordered correlation matrix on 1/31/2012.

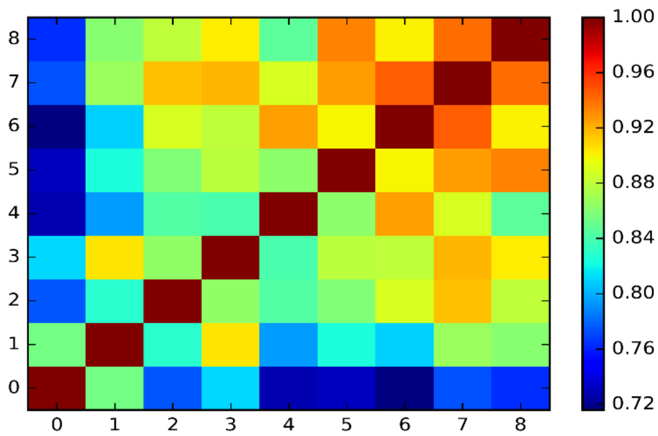

Figure 5. Heatmap of the sorted correlation matrix on 1/31/2012.

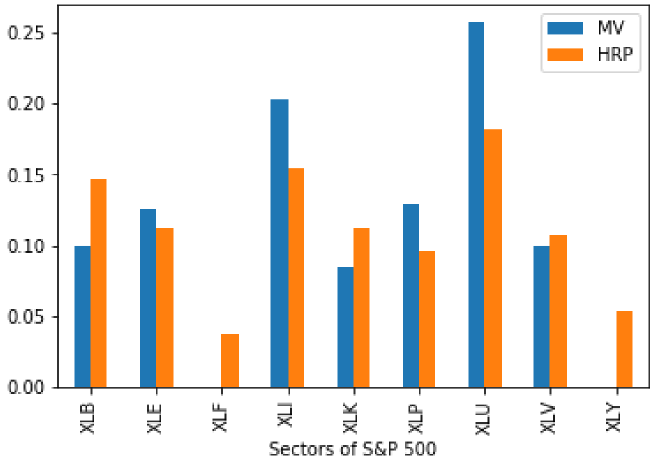

Figure 6. Weight distribution on 2/ 29/ 2000 in MV and HRP algorithm.

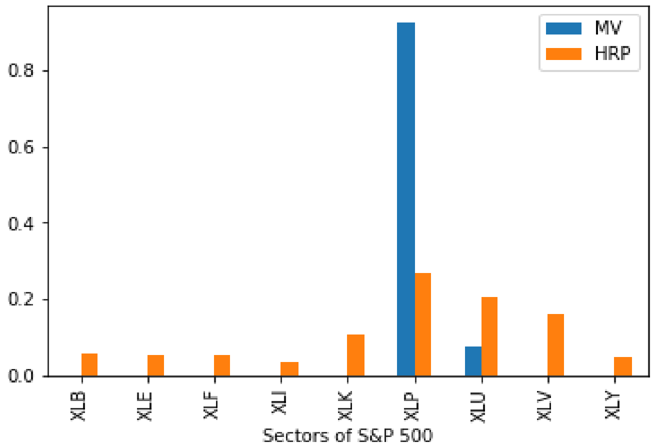

Figure 7. Weight distribution on 1/31/2012 in MV and HRP algorithm.

The stability or sensitivity of a matrix is measured by its condition number. The condition number of the covariance and correlation matrix could be determined by calculating the ratio of the absolute value between the maximum and minimum eigen values. An ill-conditioned matrix has a large condition number. In a portfolio, when the number of correlated assets increase, the condition number of the covariance matrix increases.

When we discuss the condition number of a matrix, we usually think about the sensitivity of its inverse. MV algorithm is a quadratic model and in order to calculate the weight distribution in a portfolio using equation (8), inversion of the covariance matrix is a requirement. When a matrix has a large condition number, it can be understood that the matrix is ill-conditioned or is close to being a singular matrix. An inverse of a singular matrix does not exist, and the determinant of a singular matrix is equal to zero. An inversion of the ill-conditioned matrix amplifies the estimation errors and thus resulting in a portfolio that is numerically unstable.

HRP algorithm, on the other hand, performs weight allocation based on hierarchical clustering and hence matrix inversion is not a requirement for the HRP method for performing portfolio weight distribution. A plot demonstrating the condition numbers of the covariance matrix has been presented in figure 8. It can be observed from figure 8 that the condition number of the covariance matrix changes and is high in 2012.

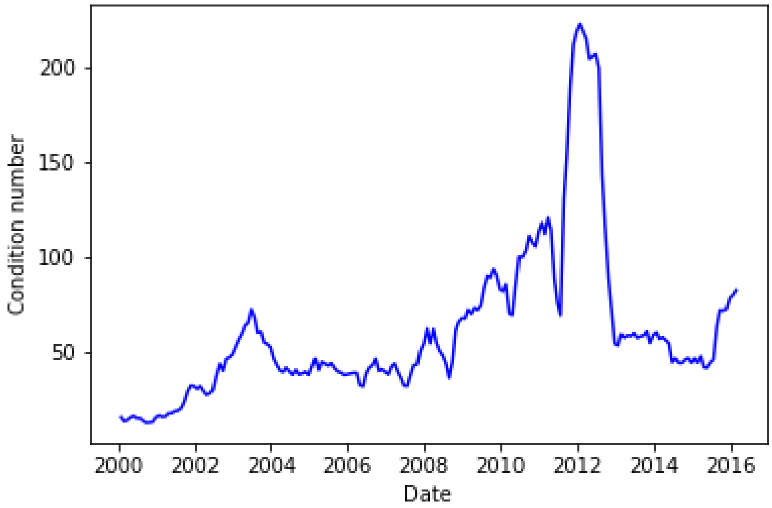

Figure 8. Variation of condition number with respect to time.

In an attempt to observe the difference in weight distribution on the assets in the two portfolios, the allocated weight on each sector of the two portfolios has been studied. Figures 9 – 17 presents a comparison of weight allocated on the nine sectors by the MV model and HRP model.

It can be observed that when the condition number started to increase, the MV model allocated weight on only a few sectors from 2000 – 2016 resulting in the concentration of risk. Weight was allocated by the MV algorithm on XLB for a very short period as can be seen from figure 9. Weight was also not allocated on XLE by the MV model from 2008 to 2014 as can be observed from figure 10.

Figure 11 and figure 12 show that for a very short time some weight was allocated by the MV model on XLF and XLI sectors respectively.

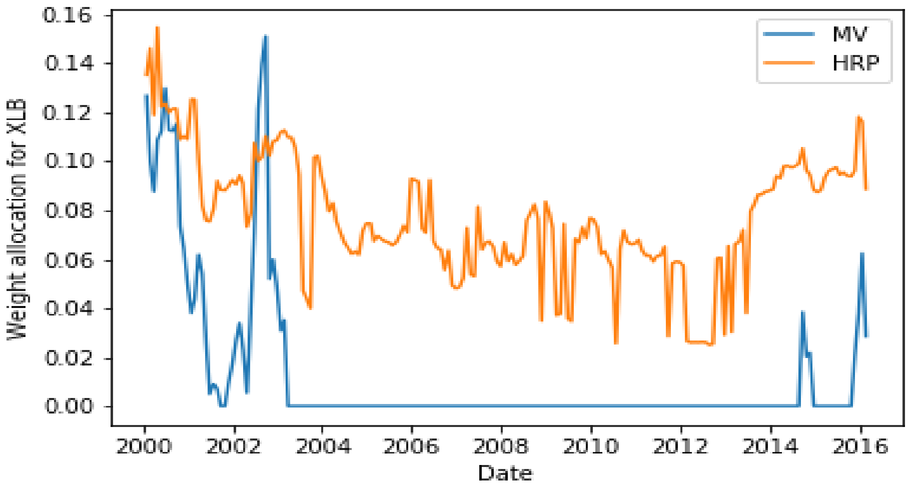

Figure 9. Weight allocation in MV model and HRP model for the XLB sector.

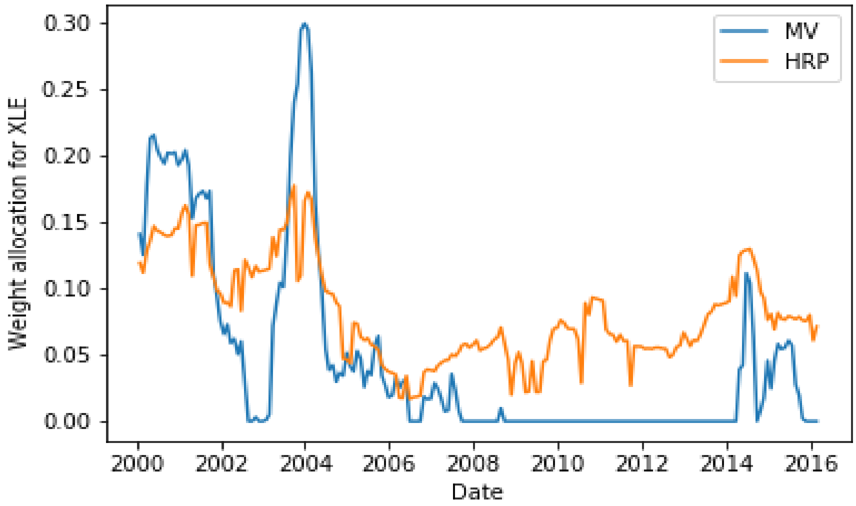

Figure 10. Weight allocation in MV and HRP model for the XLE sector.

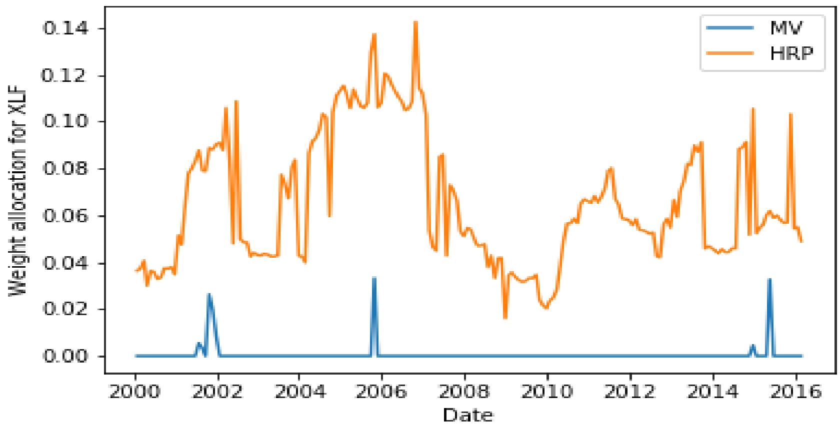

Figure 11. Weight allocation in MV and HRP model for the XLF sector.

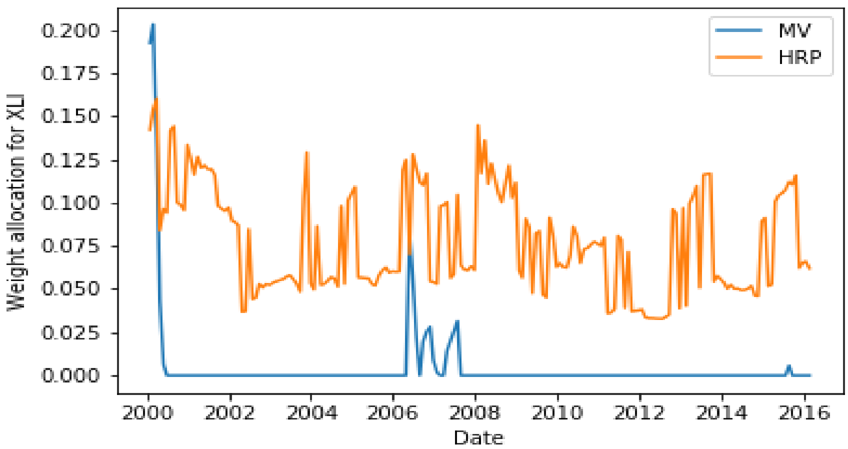

Figure 12. Weight allocation in MV and HRP model for the XLI sector.

Sporadic weight allocation by the MV algorithm can be seen on sectors XLK and XLY in figure 13 and figure 17 respectively.

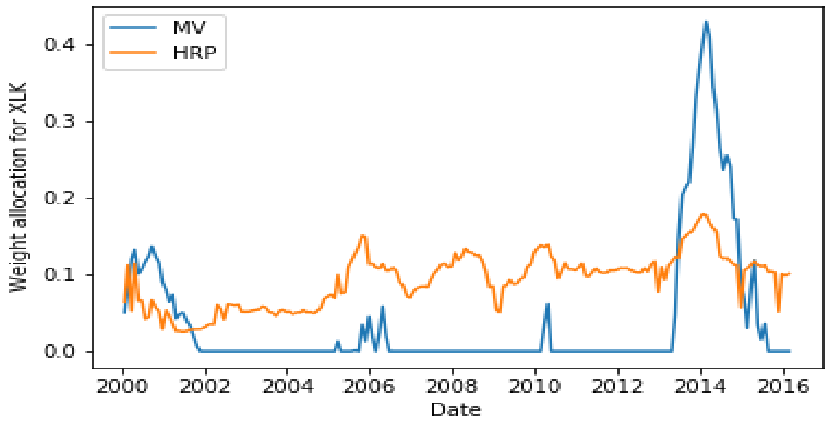

Figure 13. Weight allocation in MV and HRP model for the XLK sector.

It can be observed that the MV model from 2000 – 2016 has allocated weight only on three sectors namely, XLP as shown in figure 14, XLU as shown in figure 15, and XLV as shown in figure 16. It could be explained that the MV model being a quadratic programming method, requires an inversion of the covariance matrix for weight distribution. With the increase in the condition number from 2000 – 2016, the covariance matrix became more ill-conditioned, and inversion of an ill-conditioned matrix is prone to errors making it unreliable for practical applications. Hence, the MV model was not able to distribute weight on the sectors of the portfolio. Thus, with the increase in the condition number there appears to be a chance of risk concentration in the MV portfolio due to isolated allocation of weight among the assets from 2000 – 2016.

The HRP method, on the other hand, has been able to allocate weight on all of the nine sectors in the portfolio from 2000 – 2016 as can be observed from figures 9 – 17 and hence has been able to avoid risk concentration in the portfolio.

For portfolio weight distribution, the HRP algorithm uses hierarchical relationships between the covariance of assets and hence does not require an inversion of the covariance matrix. Thus, when the condition number of the covariance matrix started to increase and the covariance matrix became more ill-conditioned, HRP algorithm was able to successfully distribute weights among all of the assets of the portfolio and thus, chances of risk concentration was effectively reduced.

Concentration of risk in a portfolio is a potential for loss because when risk concentration takes place a group of correlated assets in the portfolio might move together in an unfavorable direction resulting in a significant loss in the portfolio. Risk concentration in a portfolio is thus undesirable and not recommended.

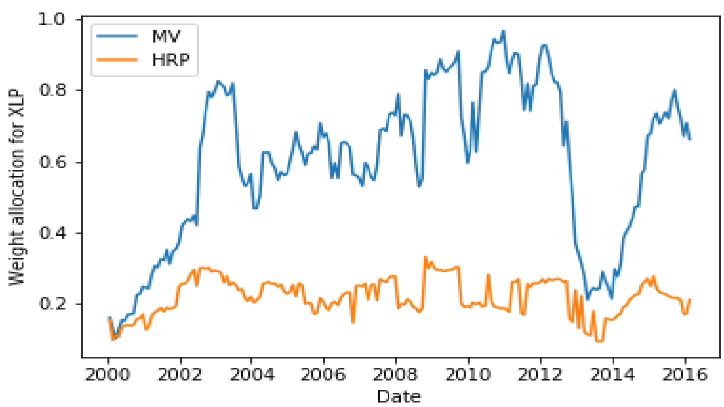

Figure 14. Weight allocation in MV and HRP model for the XLP sector.

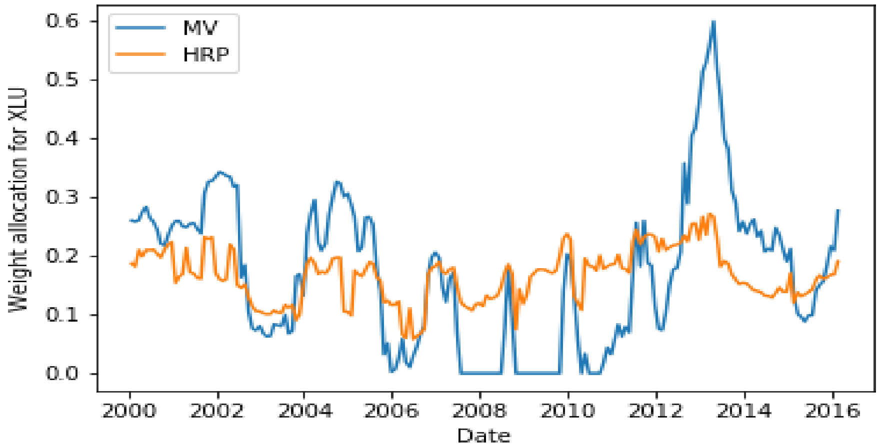

Figure 15. Weight allocation in MV model and HRP model for the XLU sector.

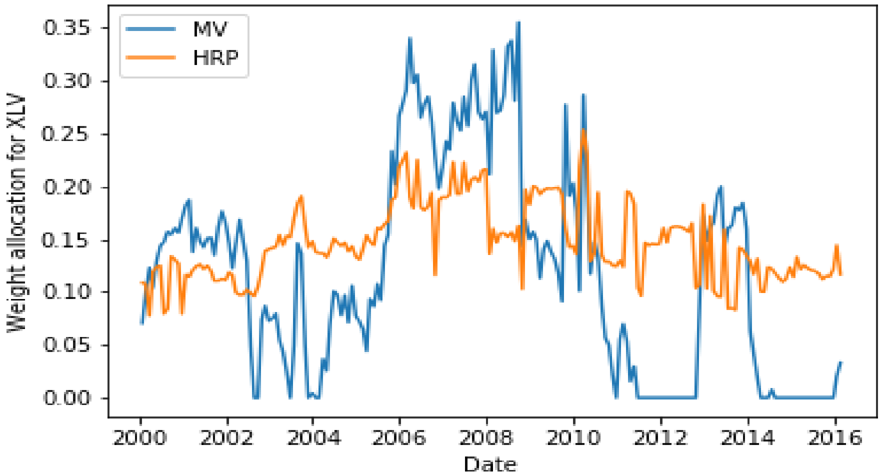

Figure 16. Weight allocation in MV model and HRP model for the XLV sector.

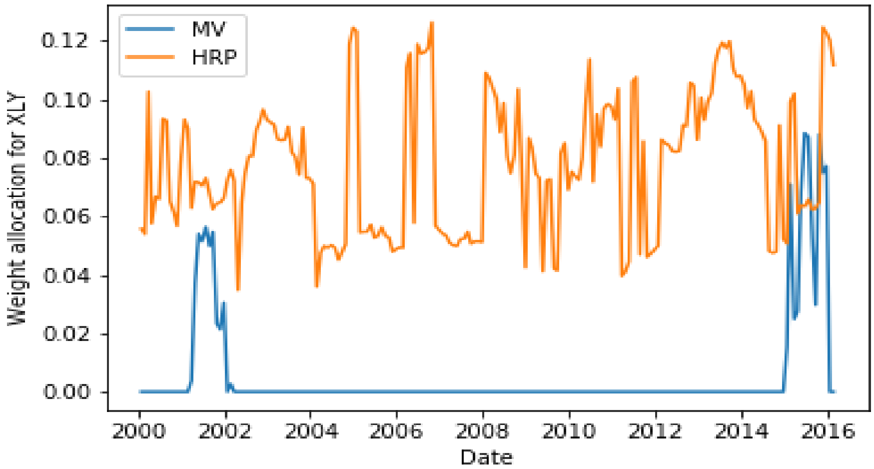

Figure 17. Weight allocation in MV model and HRP model for the XLY sector.

The daily return of the MV portfolio and the HRP portfolio has been presented in figure 18 and figure 19 respectively. It can be observed from figure 18 and figure 19 that the return for the MV and HRP portfolio is comparable.

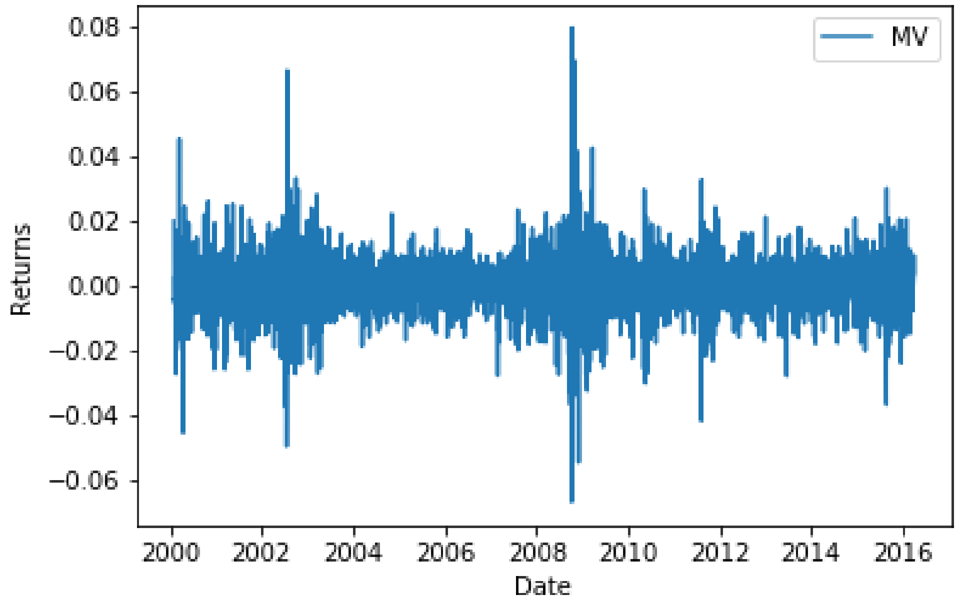

Figure 18. Daily returns of the MV portfolio.

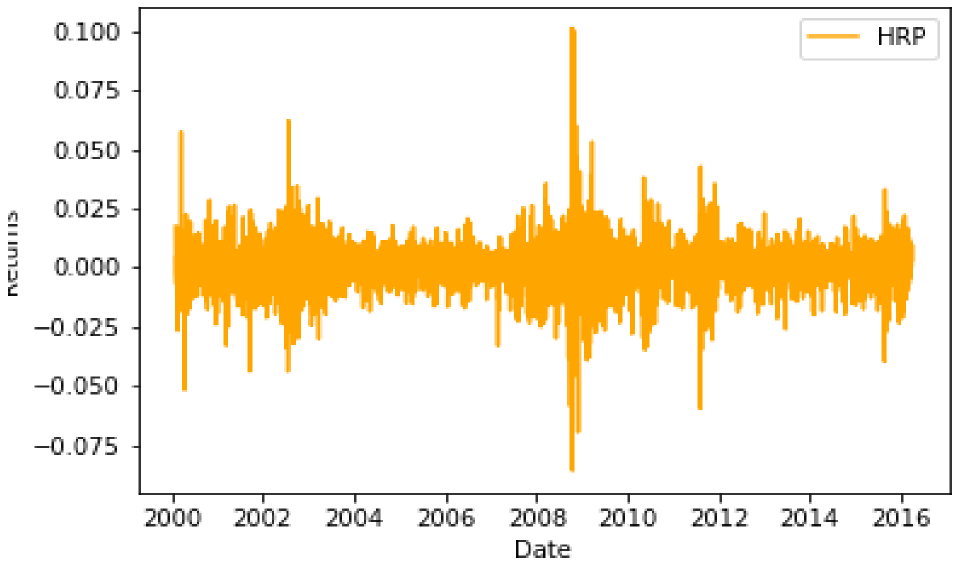

Figure 19. Daily returns of the HRP portfolio.

The cumulative return for the HRP and MV portfolio has been presented in figure 20. It can be observed from figure 20 that the cumulative return for the HRP portfolio has outperformed the MV portfolio for most of the time from 2000 – 2016.

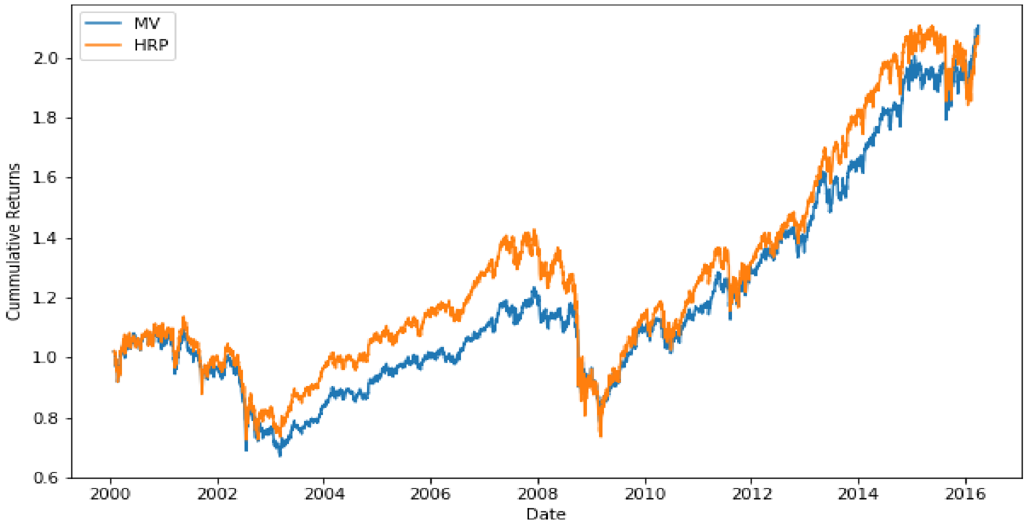

Figure 20. Cumulative returns of the MV and HRP portfolio.

The rolling volatility of the weighted MV and HRP portfolios have been calculated over an annual rolling period. The rolling volatility of the MV and HRP portfolios have been presented in figure 21.

Figure 22 presents the rolling annual return of the MV and HRP portfolio. It can be observed from figure 22 that the rolling annual return of the HRP portfolio is very similar to the rolling annual return of the MV portfolio with the HRP portfolio outperforming the MV portfolio many times over the period.

In figure 23, the performance of the MV portfolio has been compared to the performance of the HRP portfolio using Sharpe ratio. From the Sharpe ratio comparison plot it can be observed that the performance of the HRP portfolio is very comparable to the performance of the MV portfolio.

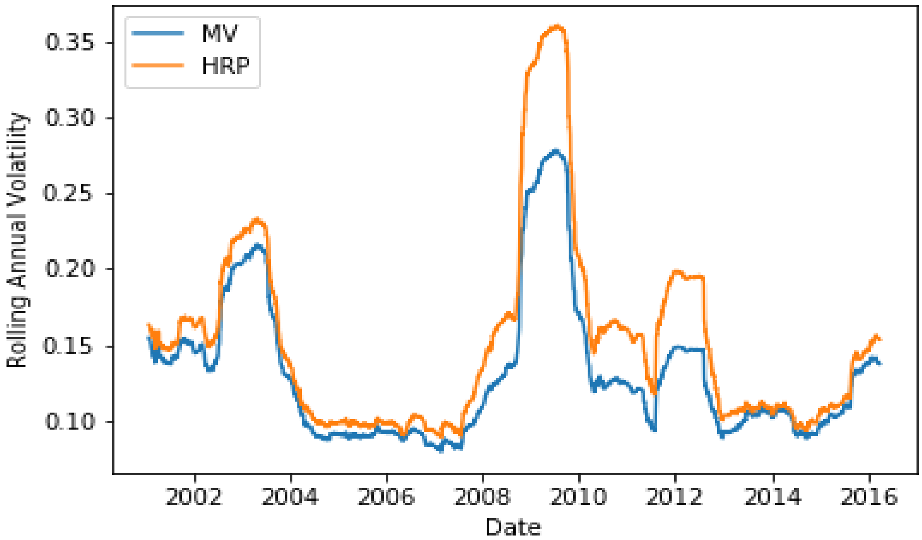

Figure 21. Rolling annual volatility for the HRP and MV portfolio.

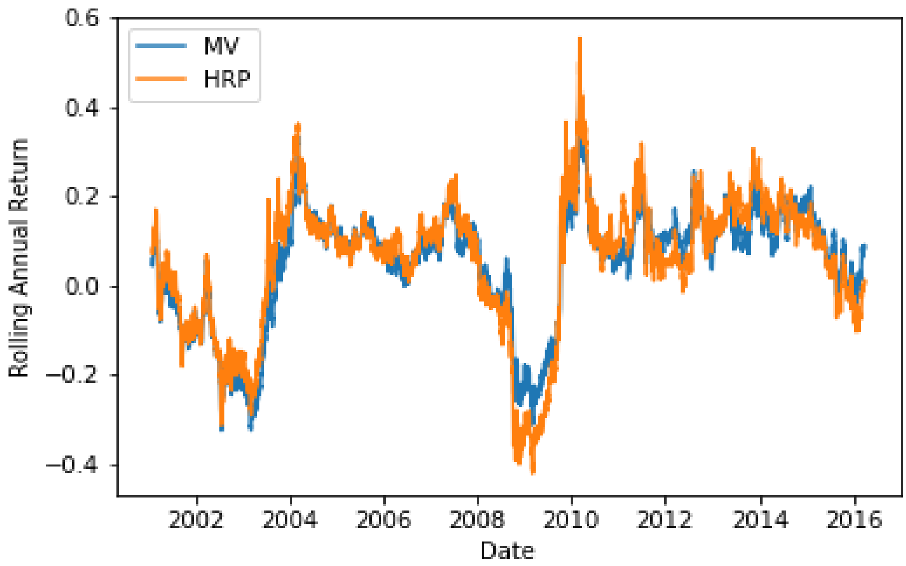

Figure 22. Rolling annual return for the HRP and MV portfolio.

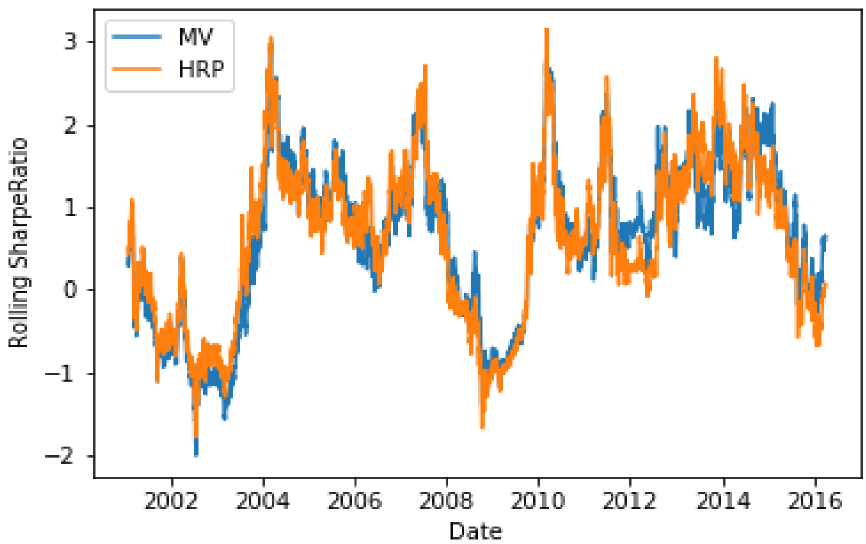

Figure 23. Rolling Sharpe Ratio based on rolling annual return and volatility.

This study aims at comparing the performance of the two portfolios created using the MV method which is a risk-based asset allocation method and HRP method which is a hierarchical clustering-based asset allocation method. The assets in the portfolio have been chosen in such a way that both MV method and HRP method will be able to distribute the weight on the portfolios. Comparing the performance of the two portfolios, it can be observed that the performance of the HRP method is comparable to the performance of the MV method with oftentimes the HRP method outperforming the MV method.

However, in practical scenarios in portfolio construction, many correlated assets may be present in a portfolio. With the increase in the number of correlated assets being added to the portfolio, the covariance matrix of the portfolio becomes more and more ill-conditioned tending towards a singular matrix. Inversion of an ill-conditioned matrix leads to errors and inversion of a singular matrix is not possible. All traditional risk-based asset allocation methods require an inversion of the covariance matrix to allocate weight in a portfolio and hence, in many practical scenarios it may not be feasible to use a traditional risk-based asset allocation method, such as the MV method.

In this study it has been shown that HRP is a feasible method in such practical situations for portfolio construction because it uses hierarchical relationship of the covariance between the assets in a portfolio. It has been shown in this analysis that the HRP method can be successfully implemented in practical portfolio constructions with US equity assets.

5. Discussion and implications

It has been demonstrated in this research that clustering-based methods can be used successfully in economic studies of portfolio construction. It has been shown in detail how a change in the condition number of the covariance matrix of a portfolio affects the allocation of weights among the assets in the portfolio for the two optimization strategies. The study shows that the HRP method can handle correlated assets in a portfolio even when the condition number is high without being challenged by risk concentration as opposed to the MV portfolio. It can be observed from this analysis that when the condition number is high, the HRP method still possesses the ability to distribute weight properly whereas a traditional risk-based asset allocation method such as the MV method has not been able to allocate weight on most of its assets under this condition.

This research asserts that HRP is a robust portfolio construction method that has a great probability of performing across various asset classes without getting challenged by the correlation between them. In practical cases of portfolio construction, presence of correlated assets is very common. The implications of this study lie in portfolio management in the field of economics. From this analysis it can be stated that the HRP method has the ability of creating and maintaining a plan for investment over the long term by meeting the financial goals in practical scenarios. In portfolio management it is very challenging to predict the performances of the asset classes. The HRP method is a machine learning technique that uses a risk parity approach to overcome this challenge by building portfolios using the risk characteristics of the assets and their correlation matrix. The method creates hierarchical structures of the invested assets using graph theory and machine learning. The hierarchical structures created in the HRP method provides a better grouping of the assets with similar characteristics into clusters without requiring an inversion of the correlation matrix. The portfolios thus created using a hierarchical clustering-based technique such as the HRP method may offer a better ability of managing the portfolio risk. It is hence being implied that machine learning methods can be used successfully in economic study of robust portfolio construction and asset allocation.

6. Conclusion

HRP and MV algorithms have been used to create two separate portfolios with the ETFs price data using 9 sectors of S&P 500. Weight has been allocated on the portfolio assets using MV model and HRP model. MV algorithm uses quadratic method (inversion of covariance matrix) for weight distribution and HRP algorithm, on the other hand, uses hierarchical clustering based on covariance of assets in the portfolio to perform portfolio weight distribution.

In quadratic equations such as the MV method, inversion of the covariance matrix is required for calculating the weight distribution. When the condition number of the covariance matrix is high, inversion of the covariance matrix leads to errors in weight allocation in the portfolio. It has been observed in this study that with the increase in the condition number from 2000 – 2016, the MV was not able to allocate weight on most of the assets in the portfolio. In the MV portfolio, out of the 9 sectors, which represent the portfolio assets, weight was allocated only on a few of the sectors, which may result in risk concentration in the MV portfolio.

On the other hand, in the HRP portfolio it has been seen in this study that weight was allocated on all the assets in the portfolio from 2000 – 2016. This was feasible for the HRP algorithm because in order to distribute the weight on the assets the HRP method does not require an inversion of the covariance matrix, which is a requirement for most of the traditional risk-based asset allocation methods. HRP method uses hierarchical relationship between the assets to allocate weight in the portfolio. Thus, in the HRP portfolio risk concentration did not occur.

Daily return of the two portfolios is comparable. HRP portfolio appears to outperform the MV portfolio in cumulative returns and rolling annual returns many times over the period from 2000 – 2016. From analyzing and comparing the rolling Sharpe ratios of the HRP and MV portfolio it can be observed that the performance of HRP is very comparable to the performance of the MV model.

The mean-variance model is dependent upon the estimation of returns. In practical scenarios, it is very difficult to estimate the market returns accurately and hence quantifying the results of the mean-variance algorithm is not always attainable. Studies have demonstrated that estimation of the expected returns has a greater influence on destabilizing the portfolios compared to the estimation of covariance matrix (Merton, 1980; Jagannathan and Ma, 2003). This research explores the performance of an alternative portfolio construction method, HRP, which does not take returns into consideration. Moreover, the mean-variance model involves inversion of the covariance matrix which might not be computationally feasible for every instance due to the presence of correlated assets in the portfolio. Besides, inversion of the covariance matrix can have a significant impact for a small change in the market correlation. HRP algorithm is not dependent upon the inversion of the covariance matrix which makes the algorithm robust, fast, and flexible. HRP is a machine learning approach that uses a hierarchical tree-clustering method to distribute the weight in a portfolio and does not use the analytical approach to calculate the weight distribution. This research shows that HRP is a feasible method for portfolio construction in the real time US equity market. HRP algorithm can allocate weights on all the assets of the portfolio irrespective of the condition number of the covariance matrix and thus chances of running into risk concentration in the portfolio gets significantly reduced. This study concludes that the HRP method can be used in practical portfolio construction to address some of the issues faced in portfolio construction using the traditional risk-based construction methods.

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Conflict of interest

The authors claim that the manuscript is completely original. The authors also declare no conflict of interest.

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