Economic Analysis Letters

Homepage: https://www.anserpress.org/journal/eal

Bayesian-Nash equilibria for fuzzy value auctions

Alexey S. Shvedov [a,*]

[a] *Department of Applied Economics, National Research University Higher School of Economics, 20 Myasnitskaya ulitsa, Moscow, 101000, Russia*

ABSTRACT

This paper analyses a model of private value auctions with symmetric risk-neutral bidders, where bidders' private values of an indivisible good are fuzzy. The auction is studied as a game with incomplete information. Fuzzy random variables, their quantile functions, and expressions for expectations through quantile functions are used. An explicit expression for the solution is found. Also, expected bidders' payments are studied.

KEYWORDS

Auction theory; Equilibrium strategy; Bidder's payment; Fuzzy random variable; Quantile function

* Corresponding author: Alexey S. Shvedov
E-mail address: ashvedov@hse.ru

ISSN 2972-3272
doi: 10.58567/eal03020008

Received 30 October 2023; Accepted 11 March 2024; Available online 13 March 2024; Version of Record 15 June 2024.

1. Introduction

Auctions are a classic and widespread procedure for selling objects. Bayesian-Nash equilibria occupy one of the central places in the auction theory (see, e.g., Krishna (2010)). Fuzzy sets provide a way to represent the factors that involve a high degree of uncertainty. The aim of this paper is to bridge the gap between Bayesian-Nash equilibria and fuzzy value auctions.

When studying private value auctions it is assumed that each bidder knows the value of the object to himself. However, in many cases, the bidder knows the value only approximately. Then, the value can be determined using membership function of a fuzzy set. For instance, a membership function can be equal to 1 on the interval [98,102], be linearly increasing from 0 to 1 on the interval [95,98], be linearly decreasing from 1 to 0 on the interval [102,107], and be equal to 0 outside the interval [95,107]. For another bidder, a membership function can be equal to 1 on the interval [94,95], be linearly increasing from 0 to 1 on the interval [92,94], be linearly decreasing from 1 to 0 on the interval [95,98], and be equal to 0 outside the interval [92,98]. These fuzzy sets are fuzzy numbers since the membership functions are defined on the set of real numbers and each membership function at first does not decrease and then does not increase. Besides, each membership function is equal to 1 at some point. In particular, a membership function can be equal to 1 at the point 97 and be equal to 0 at the remaining points; such fuzzy numbers are called crisp numbers. Then, to each fuzzy number a certain probability is assigned; the probabilities are known to all bidders. (A certain probability is assigned to each interval of fuzzy numbers if continuous distributions are used.) Thus, one should use fuzzy random variables (i.e., measurable functions whose values are fuzzy numbers) rather than random variables. The above example with fuzzy numbers may be interpreted in a way that all bidders know the value probability distribution only approximately. If there are two types of uncertainty, then fuzzy random variables may be useful. Some unknown quantities can be considered as fuzzy numbers, while others can be considered as random variables. For instance, in auctions of drilling rights, fuzzy numbers can be used to model oil tract values. However, there is another uncertainty. The bidder may be a firm alone or a joint venture. Some firms are budget-constrained. Random variables should be used to model uncertainty of this type.

This paper deals with sealed-bid auctions. A single indivisible good is to be sold to one of N bidders. A bidder with the highest bid obtains the object and has to pay a combination of the two highest bids. Each bidder knows the fuzzy value of the object to himself and does not know the value of the object to the other bidders. Each fuzzy value $\tilde{v}_i$, $i = 1,\ldots,N$, is an independent realization of a fuzzy random variable $\tilde{V}$. The quantile function of the fuzzy random variable $\tilde{V}$ is known to all bidders. For random variables, representation of probability distributions with the help of distribution functions and representation of probability distributions with the help of quantile functions are equivalent. For fuzzy random variables, quantile functions have the advantage over distribution functions (see below in Section 2).

A model of an independent private value auction was introduced by Vickrey (1961). Later, the properties of auctions with crisp values v_i have been intensely studied. Some fundamental results were obtained by Riley and Samuelson (1981), Myerson (1981), Milgrom and Weber (1982), and Cox et al. (1984). Each value v_i, $i = 1,\ldots,N$, is an independent realization of a random variable V. The distribution function F of the random variable V is known to all bidders. Let $b(v)$ be a bid, which corresponds to the value v, and the reservation price is equal to 0. It is a classical theorem that for a first-price auction a symmetric pure Bayesian-Nash equilibrium has the form

$$b(v) = v - \int_0^v \left(\frac{F(x)}{F(v)} \right)^{N-1} dx \ . \quad (1)$$

Plum (1992) proved for $N = 2$ that a similar theorem is valid for the auctions where a winner has to pay a combination of the two highest bids. Some works have dealt with the application of the fuzzy theory to auctions (see, e.g., Fang et al. (2004), Ignatius et al. (2010), Kaur et al. (2017), Zhou et al. (2021), Bhachu et al. (2023)); these works do not study Bayesian-Nash equilibria. However, Bayesian-Nash equilibria for fuzzy value auctions are of considerable interest. The existence of a distribution function F of the random variable V may reflect the fact that bidders have market statistics with regard to the competition they are facing in an auction. However, fuzzy and fuzzy random models might be preferable with not enough information. There are, also, other ways of mathematical modelling of what all bidders know the value probability distribution only approximately. In these approaches, the theory of fuzzy sets is not used. Kasberger and Schlag (2017) considered sets of conceivable environments. For any bidder, the conceivable environment consists of joint value distributions and bidding functions of other bidders. For a given conceivable environment, the bidder chooses the bidder function for which the maximum possible losses are minimized. Gretschko and Mass (2024) introduced a worst-case equilibrium which is a mixed strategy equilibrium in general. They established the existence of a worst-case equilibrium and studied the question of whether the equilibrium is unique.

In this paper, the auctions where a winner has to pay a combination of the two highest bids are studied and formula (1) is generalized in two directions. First, the theorem of Plum (1992) is generalized for arbitrary N. Second, an extension to fuzzy numbers $\tilde{v}$ is given. It is well known that the expected bidder's payment is the same for the first-price auction and the second-price auction (see, e.g., Krishna(2010)). As the present paper has shown, the expected bidder's payment for an auction with a combination of the two highest bids depends on a kind of combination in general. The paper is organized as follows. In Section 2, the model is presented. In Section 3, a theorem about bids for auctions with fuzzy values is proved and expected bidders' payments are studied.

2. Model

Suppose $r \geq 1$ is a real number. Consider a function

$$\pi(v,b,u) = \begin{cases} v - \left(r^{-1}b + (1-r^{-1})u \right) & \text{for } b > u, \\ 0 & \text{for } b \leq u, \end{cases}$$

where v, b, and u are non-negative real numbers. Suppose that v_i is a value of bidder i, b_i is a bid of bidder i, $u_i = \max_{j \neq i} b_j$, $i = 1, \ldots, N$. Then $\pi(v_i, b_i, u_i)$ is a payoff of bidder i. The auction is a first-price auction for $r = 1$ and a second-price auction for $r = \infty$. However, $r = \infty$ is not considered in this paper. For simplicity, the value of the function π is equal to 0 for $b_i = u_i$ since all probability distributions that are considered in this paper are continuous.

Let the values and the bids be fuzzy. Bidder i knows the value $\tilde{v}_i$ and does not know the values $\tilde{v}_j$, $j \neq i$, where $\tilde{v}_i$ and $\tilde{v}_j$ are fuzzy numbers. The values $\tilde{v}_i$, $i = 1, \ldots, N$, are realizations of fuzzy random variables

$\tilde{V}_i$, $i = 1,\dots,N$. The fuzzy random variables $\tilde{V}_1,\dots,\tilde{V}_N$ are independent and have the same quantile function (see Shvedov (2016b)). The quantile function is known to all bidders. By $\tilde{b}_i$ denote the bid of bidder i.

A fuzzy number $\tilde{x}$ is a subset of $\mathbb{R}^2$. The subset is determined by two functions $x^L : [0,1] \to \mathbb{R}$ and $x^R : [0,1] \to \mathbb{R}$. The functions x^L and x^R are called left index and right index of the fuzzy number $\tilde{x}$, respectively. Both functions x^L and x^R are left-continuous. The function x^L is monotone non-decreasing. The function x^R is monotone non-increasing. Furthermore, $x^L(1) \le x^R(1)$. The subset $\tilde{x}$ contains the points (ξ,η) such that $0 \le \eta \le 1$ and $x^L(\eta) \le \xi \le x^R(\eta)$ for any $\eta \in [0,1]$.

Let (Ω, F, P) be a probability space. Suppose M is a set, whose elements are compact subsets of $\mathbb{R}^2$. It is known that M is a metric space with respect to the Hausdorff distance. A function $\tilde{X} : \Omega \to \mathrm{M}$ is measurable if $\tilde{X}^{-1}(M) \in \mathrm{F}$ for all Borel sets $M \subseteq \mathrm{M}$. Suppose that $\tilde{X}(\omega)$ is a fuzzy number for any $\omega \in \Omega$ and supports of all these fuzzy numbers belong to a bounded subset of $\mathbb{R}$. Then a measurable function $\tilde{X}$ is called a fuzzy random variable.

Let $\tilde{X}$ be a fuzzy random variable. Suppose $\tilde{X}(\omega) = \tilde{x}$, x^L and x^R are left and right indices of the fuzzy number $\tilde{x}$, respectively. Consider functions $X^L(\omega,\eta) = x^L(\eta)$ and $X^R(\omega,\eta) = x^R(\eta)$. Then $X^L(\eta) : \Omega \to \mathbb{R}$ and $X^R(\eta) : \Omega \to \mathbb{R}$ for any $\eta \in [0,1]$. It is proved by Shvedov (2016a) that $X^L(\eta)$ and $X^R(\eta)$ are random variables for any $\eta \in [0,1]$. Sets of random variables $X^L(\eta)$ and $X^R(\eta)$, $\eta \in [0,1]$, are called left index and right index of the fuzzy random variable $\tilde{X}$, respectively.

The above definition of fuzzy random variables is a modification of a well-known definition of fuzzy random variables (Kwakernaak (1978), Kwakernaak (1979), Puri and Ralescu (1986)). The modification is necessary to define quantile function of a fuzzy random variable.

By F denote a distribution function of a random variable X. Assume that any distribution function that is considered in this paper is monotone increasing on a segment $[0,\alpha]$, continuously differentiable inside $[0,\alpha]$, $F(0) = 0$, and $F(\alpha) = 1$. Though, segments $[0,\alpha]$ may be different for different functions F. Quantile function of the random variable X is defined as $q(p) = F^{-1}(p)$, $0 < p < 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Borel function. Suppose that expectation of the random variable $\varphi(X)$ exists. Then (see Shvedov (2016b))

$$E(\varphi(X)) = \int_0^1 \varphi(q(p))dp \ . \tag{2}$$

Formula (2) is the foundation of further proof.

In accordance with the definition from Shvedov (2016b), the quantile function of the fuzzy random variable $\tilde{V}$ is a family of fuzzy numbers $\tilde{q}(p)$, $0 < p < 1$. Quantile function of a fuzzy random variable possesses an advantage over distribution function of a fuzzy random variable since the quantile function is a function of a real argument, not a fuzzy argument. Impose a restriction to the quantile function of the fuzzy random variable $\tilde{V}$. Suppose that there is a fuzzy number $\tilde{a}$ such that $\tilde{V} = V\tilde{a}$. Then

$$\tilde{q}(p) = q_0(p)\tilde{a}, \qquad 0 < p < 1,$$

where q_0 is quantile function of the random variable V. Let $a^L(\eta)$ and $a^R(\eta)$, $\eta \in [0,1]$, be left index and right index of the fuzzy number $\tilde{a}$, respectively. Assume that $a^L(0) > 0$. Let $V_i^L(\eta)$ and $V_i^R(\eta)$, $\eta \in [0,1]$, be left index and right index of the fuzzy random variable $\tilde{V}_i$, respectively. It is possible to drop i since the fuzzy random variables $\tilde{V}_i$ have the same quantile function. Then quantile function of the random variable $V^L(\eta)$ has the form $a^L(\eta)q_0(p)$ and quantile function of the random variable $V^R(\eta)$ has the form $a^R(\eta)q_0(p)$.

A strategy is a function H such that $H(\tilde{v}) = \tilde{b}$, where $\tilde{v}$ is a value and $\tilde{b}$ is a bid. Impose a restriction to the function H. Suppose that there is a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$b^L(\eta) = h(v^L(\eta)), \qquad b^R(\eta) = h(v^R(\eta))$$

for all $\eta \in [0,1]$. In addition, suppose that the function h is monotone increasing, continuously differentiable, and $h(x) \to 0$ as $x \to 0$.

Consider random variables

$$U_i^L(\eta) = \max_{j \neq i} V_j^L(\eta), \qquad U_i^R(\eta) = \max_{j \neq i} V_j^R(\eta) \tag{3}$$

for all $i = 1, \dots, N$ and for all $\eta \in [0,1]$. Denote

$$\Pi_i^L(v,b,h,\eta) = E(\pi(v,b,h(U_i^L(\eta)))), \qquad \Pi_i^R(v,b,h,\eta) = E(\pi(v,b,h(U_i^R(\eta)))), \tag{4}$$

where v and b are non-negative real numbers, $\eta \in [0,1]$. Let $v_i^L(\eta)$ and $v_i^R(\eta)$, $\eta \in [0,1]$, be left index and right index of the fuzzy number $\tilde{v}_i$, respectively. Let $b_i^L(\eta)$ and $b_i^R(\eta)$, $\eta \in [0,1]$, be left index and right index of the fuzzy number $\tilde{b}_i$, respectively.

A strategy h is called a symmetric pure Bayesian-Nash equilibrium if the following conditions hold:

$$\Pi_i^L(v_i^L(\eta), h(v_i^L(\eta)), h, \eta) = \max_{b_i^L(\eta)} \Pi_i^L(v_i^L(\eta), b_i^L(\eta), h, \eta),$$

$$\Pi_i^R(v_i^R(\eta), h(v_i^R(\eta)), h, \eta) = \max_{b_i^R(\eta)} \Pi_i^R(v_i^R(\eta), b_i^R(\eta), h, \eta)$$

for all $i = 1, \ldots, N$, for all $\tilde{v}_i$, and for all $\eta \in [0,1]$.

3. Main result

Denote

$$c_p = q_0(p) - \frac{1}{p^{r(N-1)}} \int_0^{q_0(p)} F_0(x)^{r(N-1)} dx, \qquad 0 < p < 1,$$

where F_0 is the distribution function that corresponds to the quantile function q_0.

Theorem. Let $\tilde{v} = q_0(p)\tilde{a}$ be a value. Then the equilibrium strategy $\tilde{b} = H(\tilde{v})$ has the form $\tilde{b} = c_p \tilde{a}$.

Remark. Suppose that $a^L(\eta) = a^R(\eta) = 1$ for all $\eta \in [0,1]$. This means that we consider the crisp problem. Then the equilibrium strategy becomes

$$b(v) = v - \int_0^v \left(\frac{F(x)}{F(v)} \right)^{r(N-1)} dx. \qquad (5)$$

Formula (5) is a generalization of formula (1). For $N = 2$, formula (5) is derived by Plum (1992). However, the method of Plum (1992) is a more sophisticated method rather than our method. It seems that the method of Plum (1992) is suitable only for $N = 2$.

Proof of the theorem. Denote $U = U_i^L(\eta)$. Let Q be quantile function of the random variable U, Φ be distribution function of the random variable U, and u be a realization of the random variable U. Denote $\Pi(u,b,h) = E\left(\pi(u,b,h(U))\right)$. Suppose that a function h satisfies the following property:

$$\Pi(u, h(u), h) = \max_b \Pi(u, b, h)$$

for any value u. Let us show that

$$h(z) = z - \frac{1}{\Phi(z)^r} \int_0^z \Phi(t)^r dt. \qquad (6)$$

Consider a function

$$s(b)=Q^{-1}\left(h^{-1}(b)\right). \tag{7}$$

Then the inequality $b>h(Q(p))$ is equivalent to the inequality $p<s(b)$. It follows from (2) and (4) that

$$\Pi(v,b,h)=\int_0^{s(b)}\left(v-\left(r^{-1}b+(1-r^{-1})h(Q(p))\right)\right)dp.$$

We have

$$\frac{d}{db}\Pi(v,b,h)=\int_0^{s(b)}\frac{d}{db}\left(v-\left(r^{-1}b+(1-r^{-1})h(Q(p))\right)\right)dp+$$

$$+s'(b)\left(v-\left(r^{-1}b+(1-r^{-1})h(Q(s(b)))\right)\right)=-r^{-1}s(b)+s'(b)(v-b).$$

The first-order condition becomes

$$(v-b)s'(b)-r^{-1}s(b)=0.$$

It follows from the symmetry of the bidders that $v=h^{-1}(b)$. Thus,

$$(h^{-1}(b)-b)s'(b)-r^{-1}s(b)=0. \tag{8}$$

By using (7) we get

$$s'(b)=\Phi'\left(h^{-1}(b)\right)\left(h^{-1}\right)'(b). \tag{9}$$

Denote $z=h^{-1}(b)$. Differentiating $z=h^{-1}(h(z))$ we obtain

$$1=\left(h^{-1}\right)'(h(z))h'(z).$$

Thus, it follows from (9) that

$$s'(b)=\frac{\Phi'(z)}{h'(z)}.$$

Therefore, (8) becomes

$$(z-h(z))\frac{\Phi'(z)}{h'(z)}=r^{-1}\Phi(z). \tag{10}$$

The solution of the equation

$$y'+f(x)y=g(x)$$

with an initial condition $y(x_0)=y_0$ has the form

$$y(x)=e^{-G(x)}\left(y_0+\int_{x_0}^{x}g(t)e^{G(t)}dt\right),$$

where

$$G(x)=\int_{x_0}^{x}f(t)dt.$$

Hence, the solution of the equation (10) with the initial condition $h(0)=0$ is (6).

Suppose that the function h is defined by formula (6). Then

$$\Pi(v,h(z),h)=\int_0^{s(h(z))}\left(v-\left(r^{-1}h(z)+(1-r^{-1})h(Q(p))\right)\right)dp.$$

By (6) and (7) it follows that

$$\Pi(v,h(z),h)=\Phi(z)v-r^{-1}\Phi(z)\left(z-\frac{1}{\Phi(z)^r}\int_0^z\Phi(t)^r\,dt\right)-$$

$$-(1-r^{-1})\int_0^{\Phi(z)}\left(Q(p)-\frac{1}{\Phi(Q(p))^r}\int_0^{Q(p)}\Phi(t)^r\,dt\right)dp.$$

By straightforward transformation one can get

$$\frac{d}{dz}\Pi(v,h(z),h)=\Phi'(z)(v-z).$$

This means that the function $\Pi(v,h(z),h)$, which is considered as a function of the variable z, has a maximum at $z=v$. Thus, the function h is an equilibrium strategy for the crisp case.

The random variables $V_1^L(\eta),\ldots,V_N^L(\eta)$ are independent and identically distributed. Denote by $F^L(z;\eta)$ distribution function of the random variable $V_j^L(\eta)$. It follows from (\ref{a1a}) that $\Phi(z)=F^L(z;\eta)^{N-1}$. By using (6) we get

$$h(v^L(\eta))=v^L(\eta)-\frac{1}{F^L(v^L(\eta);\eta)^{r(N-1)}}\int_0^{v^L(\eta)}F^L(t;\eta)^{r(N-1)}\,dt. \tag{11}$$

By (11) it follows that

$$b^L(\eta)=h(v^L(\eta))=a^L(\eta)q_0(p)-\frac{1}{F_0(q_0(p))^{r(N-1)}}\int_0^{a^L(\eta)q_0(p)}F_0\left(\frac{t}{a^L(\eta)}\right)^{r(N-1)}dt=$$

$$=a^L(\eta)\left(q_0(p)-\frac{1}{p^{r(N-1)}}\int_0^{q_0(p)}F_0(u)^{r(N-1)}\,du\right)=a^L(\eta)c_p,$$

where $0\le\eta\le1$. Similarly, $b^R(\eta)=a^R(\eta)c_p$.

This completes the proof of the theorem.

For fixed $\tilde{v}$, $\tilde{b}=H(\tilde{v})$ is a fuzzy number. Risk aversion of a bidder can be expressed by the method of defuzzification.

Example 1. Suppose $r=1$, $N=2$, $q_0(p)=p$. Then $F_0(x)=x$ for $0\le x\le1$. By definition

$$c_p=q_0(p)-\frac{1}{p}\int_0^{q_0(p)}F_0(x)dx=\frac{1}{2}q_0(p).$$

By the Theorem $\tilde{b} = 0.5\,\tilde{v}$. Assume that defuzzification of the fuzzy number $\tilde{a}$ is carried out using the following formula

$$\int_0^1 \left(\lambda a^L(\eta) + (1-\lambda)a^R(\eta)\right) d\eta, \quad 0 \le \lambda \le 1.$$

A bidder's level of risk aversion is determined by λ. Suppose that $a^L(\eta) = 90 + 10\eta$, $a^R(\eta) = 110 - 10\eta$, $\tilde{v}_1 = 0.76\,\tilde{a}$, $\tilde{v}_2 = 0.72\,\tilde{a}$. Then $\tilde{b}_1 = 0.38\,\tilde{a}$, $\tilde{b}_2 = 0.36\,\tilde{a}$. The first bidder is risk averse and chooses $\lambda = 1$; his bid is equal to 36.1. The second bidder is not risk averse and chooses $\lambda = 0$; her bid is equal to 37.8. Therefore, the bid depends on both the bidder's value and the bidder's level of risk aversion.

Example 2. Using (5), we get $b(0) = 0$, $b^{'}(v) > 0$ for $v > 0$. Let v be the value of the bidder i. Then the payment of the bidder i is

$$\tau(U_i, v) = \begin{cases} 0 & \text{for } U_i > v, \\ r^{-1}b(v) + (1-r^{-1})b(U_i) & \text{for } U_i \le v. \end{cases}$$

Let $G_0(v) = F_0(v)^{N-1}$ be distribution function of the random variable U_i. Then the expected payment of the bidder i is

$$E\left(\tau(U_i, v)\right) = \frac{1}{r} b(v) G_0(v) + \left(1 - \frac{1}{r}\right) \int_0^v b(x) dG_0(x). \tag{12}$$

For fixed $v > 0$, consider the function

$$G(x) = \begin{cases} 0 & \text{for } x = 0, \\ 0.5 & \text{for } 0 < x < v, \\ 1 & \text{for } x = v. \end{cases}$$

Using the function $G(x)$ instead of the function $G_0(x)$, from (12) and (5), we obtain

$$E\left(\tau(U_i, v)\right) = \frac{1}{r} b(v) + \left(1 - \frac{1}{r}\right) 0.5 b(v) = 0.5\left(1 + \frac{1}{r}\right)\left(1 - \frac{1}{2^r}\right) v.$$

One can find a smooth and strictly increasing function $G_0(x)$ for which $G_0(0) = 0$, $G_0(v) = 1$, in such a way that the integral $\int_0^v b(x) dG_0(x)$ is arbitrary close to the integral $\int_0^v b(x) dG(x)$. Thus, the expected payment $E\left(\tau(U_i, v)\right)$ in general depends on r. However, in the present case the expected payment is equal to \$0.5 v\$ both for $r = 1$ and $r = \infty$.

4. Conclusion

The paper presents one of the possible approaches to combined use of stochastic and fuzzy methods in the auction theory. A combination of stochastic and fuzzy methods has proved useful in modelling diverse types of economic problems. This combination enables to more subtly handle the information received. For example, Shvedov (2023) studied Cournot equilibrium under uncertain yield; the fuzzy random approach allows to get rid of the drawbacks that exist at the random approach. Fuzzy random variables can be used to study auctions with interdependent values. This may be a line of future research.

Funding Statement

This research received no external funding.

Conflict of interest

The author claims that the manuscript is completely original. The author also declares no conflict of interest.

Acknowledgments

Acknowledgments to anonymous referees' comments and editor's effort.

References

Bhachu, K., Elkasrawy, A., and Venkatesh B. (2023) Fuzzy Optimization Model of an Incremental Capacity Auction Formulation with Greenhouse Gas Consideration. *IET Smart Grid* 6, 124-135. https://doi.org/10.1049/stg2.12065

Cox, J., Smith, V., and Walker, J. (1984). Theory and Behavior of Multiple Unit Discriminative Auctions. *Journal of Finance* 39, 983-1010. https://doi.org/10.1111/j.1540-6261.1984.tb03888.x

Fang, S., Nuttle, H., and Wang, D. (2004). Fuzzy Formulation of Auctions and Optimal Sequencing for Multiple Auctions. *Fuzzy Sets and Systems* 142, 421-441. https://doi.org/10.1016/S0165-0114(03)00127-1

Gretscko, V., and Mass, H. (2024). Worst-Case Equilibria in First-Price Auctions. *Theoretical Economics*. 19, 61-93. https://doi.org/10.3982/TE4555

Ignatius, J., Motlagh, S., Sepheri, M., Lai, Y., Mustafa, A. (2010). Fuzzy Prices in Combinatorial Auctions. In: L.C. Jain and C.P. Lim (Eds.): Handbook on Decision Making, ISRL 4, Berlin, Springer, 347-367. https://doi.org/10.1007/978-3-642-13639-9_14

Kasberger, B., and Schlag, K. (2017). Robust Bidding in First-Price Auctions: How to Bid Without Knowing what Others are Doing. Vienna Economics Papers vie1707, University of Vienna, Department of Economics. https://papersecon.univie.ac.at/RePEc/vie/viennp/vie1707.pdf

Kaur, P., Goyal, M., and Lu, J. (2017). A Comparison of Bidding Strategies for Online Auctions Using Fuzzy Reasoning and Negotiation Decision Functions. *IEEE Transactions on Fuzzy Systems* 25, 425-438. https://doi.org/10.1109/TFUZZ.2016.2598297

Krishna, V. (2010). Auction Theory, 2nd ed., San Diego, Elsevier / Academic Press.

Kwakernaak, H. (1978). Fuzzy Random Variables - I. Definitions and Theorems. *Information Sciences* 15, 1-29. https://doi.org/10.1016/0020-0255(78)90019-1

Kwakernaak, H. (1979). Fuzzy Random Variables - II. Algorithms and Examples for the Discrete Case. *Information Sciences* 17, 153-178. https://doi.org/10.1016/0020-0255(79)90020-3

Milgrom, P., and Weber R. (1982). A Theory of Auctions and Competitive Bidding. *Econometrica* 50, 1089-1122. https://doi.org/0012-9682(198209)50:5%3C1089:ATOAAC%3E2.0.CO;2-E

Myerson, R. (1981). Optimal Auction Design. *Mathematics of Operations Research* 6, 58-73.

Plum, M. (1992). Characterization and Computation of Nash-Equilibria for Auctions with Incomplete Information. *International Journal of Game Theory* 20, 393-418. https://doi.org/10.1007/BF01271133

Puri, M. and, Ralescu D. (1986). Fuzzy Random Variables. *Journal of Mathematical Analysis and Applications* 114, 409-422. https://doi.org/10.1016/0022-247X(86)90093-4

Riley, J., and Samuelson W. (1981). Optimal Auctions. *American Economic Review* 71, 381-392.

Shvedov, A. (2016a). Estimating the Means and the Covariances of Fuzzy Random Variables. *Applied Econometrics* 42, 121-138. (In Russ.) http://pe.cemi.rssi.ru/pe_2016_42_121-138.pdf
Shvedov, A. (2016b). Quantile Function of a Fuzzy Random Variable and an Expression for Expectations. *Mathematical Notes* 100, 477-481. https://doi.org/10.1134/S0001434616090157
Shvedov, A. (2023). Cournot Equilibrium Under Fuzzy Random Yield. *HSE Economic Journal* 27, 435-448. (In Russ.) https://doi.org/10.17323/1813-8691-2023-27-3-435-448
https://ej.hse.ru/data/2023/10/23/2049770577/Шведов.pdf
Vickrey, W. (1961). Counterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance* 16, 3-37. https://doi.org/10.1111/j.1540-6261.1961.tb02789.x
Zhou, K., Ma, G., Wang, Y., Zhong, J., Wang, S., and Tang Y. (2021). Strategy Research of Used Cars in Online Sequential Auction Based on Fuzzy Theory. *Journal of Intelligent and Fuzzy Systems* 40, 4967-4977. https://doi.org/10.3233/JIFS-201719